

THE EQUIVARIANT COHOMOLOGY OF COMPLEXITY ONE SPACES

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ABSTRACT. Complexity one spaces are an important class of examples in symplectic geometry. Karshon and Tolman classify them in terms of combinatorial and topological data. In this paper, we compute the equivariant cohomology for any complexity one space $T^{n-1} \curvearrowright M^{2n}$. The key step is to compute the equivariant cohomology for any Hamiltonian $S^1 \curvearrowright M^4$.

1. INTRODUCTION

An effective symplectic action of a torus $T = (S^1)^k$ on a symplectic manifold (M, ω) is called **Hamiltonian** if it admits a **momentum map**, a smooth map $\Phi: M \rightarrow \mathfrak{t}^* \simeq \mathbb{R}^k$ such that $d\Phi_j = -\iota(\xi_j)\omega$ for all $j = 1, \dots, k$, where ξ_1, \dots, ξ_k are the vector fields that generate the torus action. The vector fields ξ_1, \dots, ξ_k define an isotropic subbundle of the tangent bundle, so we must have $\dim(T) \leq \frac{1}{2} \dim(M)$. When M is two-dimensional, the only example of a Hamiltonian action is a linear action $S^1 \curvearrowright S^2$. When M is four-dimensional, the only tori that act effectively are S^1 and $(S^1)^2$. When we have the equality $\dim(T) = \frac{1}{2} \dim(M)$, the action is called **toric**. A **complexity one space** is a symplectic manifold equipped with a Hamiltonian action of a torus which is one dimension less than half the dimension of the manifold.

Equivariant cohomology is a generalized cohomology theory in the equivariant category. In general, Hamiltonian T -spaces enjoy a number of useful features when it comes to computing equivariant cohomology. The set M^T of fixed points plays a leading role. For a Hamiltonian action $S^1 \curvearrowright M^4$ with only isolated fixed points, Goldin and the first author [5] use the Atiyah-Bott/Berline-Vergne (ABBV) localization formula [1, 2] to describe the equivariant cohomology $H_{S^1}^*(M; \mathbb{Q})$. In this case, the S^1 -action extends to a toric action $T^2 \curvearrowright M^4$. In general, a Hamiltonian S^1 -action on a four-manifold might fix two-dimensional submanifolds, and it need not extend to a toric action.

The first main result of this manuscript describes the S^1 -equivariant cohomology for any Hamiltonian S^1 -action on a symplectic four-manifold. Examples include k -fold blowups of symplectic ruled surfaces of positive genus. This is a rare instance in the symplectic category where the presence of odd degree cohomology doesn't make calculations in equivariant cohomology impossible. It is also the first occurrence of calculations with fixed point components of different diffeomorphism types.

1.1. Theorem. *Let M^4 be a compact connected symplectic four-manifold.*

(A) *Let $S^1 \curvearrowright M^4$ be a Hamiltonian circle action. The inclusion $i : M^{S^1} \hookrightarrow M$ induces an injection in integral equivariant cohomology*

$$i^* : H_{S^1}^*(M; \mathbb{Z}) \hookrightarrow H_{S^1}^*(M^{S^1}; \mathbb{Z}).$$

(B) *In equivariant cohomology with rational coefficients, the image of i^* is characterized as those classes $\alpha \in H_{S^1}^*(M^{S^1}; \mathbb{Q}) = \bigoplus_{F \subset M^{S^1}} H_{S^1}^*(F; \mathbb{Q})$ which satisfy:*

- (0) *that the degree zero terms $\alpha^{(0)}|_F$ are all equal;*
- (1) *that the degree one terms $\alpha^{(1)}|_\Sigma$ restricted to fixed surfaces are equal; and*
- (2) *the ABBV relation*

$$(1.2) \quad \sum_{F \subset M^{S^1}} \pi_*^F \left(\frac{\alpha|_F}{e_{S^1}(\nu(F \subseteq M))} \right) \in \mathbb{Q}[u] = H_{S^1}^*(pt; \mathbb{Q}),$$

where the sum is taken over the connected components F of the fixed point set M^{S^1} , $\alpha|_F$ is the restriction of α to the component F , and π_*^F is the equivariant pushforward map.

The proof boils down to homological algebra. The S^1 -momentum map is a perfect Morse-Bott function with critical set M^{S^1} , so we will use Morse theory to compute the equivariant Poincaré polynomials $P_{S^1}^{M^{S^1}}(t)$ and $P_{S^1}^M(t)$ and their difference. This tells us the ranks of $i^*(H_{S^1}^*(M; \mathbb{Q}))$. On the other hand, we can also determine the ranks of the submodule of $H_{S^1}^*(M^{S^1}; \mathbb{Q})$ the requirements in Theorem 1.1 cut out. It will then be straight forward to check that the two lists of ranks agree. We can check directly that tuples $(\alpha|_F)$ in the image of i^* must satisfy items (0) and (1); and (2) follows from the Atiyah-Bott/Berline-Vergne localization formula described in Theorem 3.5. We conclude that the two submodules are equal. We show a sample class satisfying the ABBV relations in Figure 1.3.

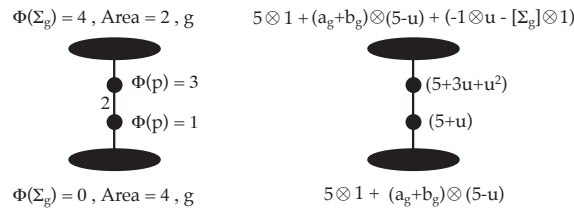


FIGURE 1.3. On the left is the decorated graph, and on the right a collection of classes in $H_{S^1}^*(F)$ for each fixed component F . These classes satisfy the requirements in Theorem 1.1, so they are the restrictions to the fixed sets of a global class in $H_{S^1}^*(M; \mathbb{Q})$.

Our paper is organized as follows. We review the combinatorial data associated to Hamiltonian $S^1 \curvearrowright M^4$ in Section 2, and we give an overview of equivariant cohomology

in Section 3. The technical heart of the paper is Section 4, where we prove Theorem 1.1. Then we will use a theorem of Tolman and Weitsman [12] in Section 5 to assemble the equivariant cohomology of a complexity one space, $T^{n-1} \curvearrowright M^{2n}$, in Corollary 5.1. This result demonstrates how amenable complexity one spaces are to algebraic computation. It opens the door to questions about the geometric data encoded in the equivariant cohomology ring for complexity one spaces, along the lines of Masuda's work [11].

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2. HAMILTONIAN CIRCLE ACTIONS AND DECORATED GRAPHS

Let (M^4, ω) be a symplectic four-manifold with an effective Hamiltonian S^1 -action. The real-valued momentum map $\Phi : M \rightarrow \mathbb{R}$ is a Morse-Bott function with critical set corresponding to the fixed points. Since M is four-dimensional, the critical set can only consist of isolated points and two-dimensional submanifolds. The latter can only occur at the extrema of Φ . The maximum and minimum of the momentum map is each attained on exactly one component of the fixed point set. This is the key point for our computations below. We call the triple (M, ω, Φ) a **Hamiltonian S^1 -space**.

To (M, ω, Φ) Karshon associates the following **decorated graph** [8, §2.1]. For each isolated fixed point p there is a vertex $\langle p \rangle$, labeled by the real number $\Phi(p)$. For each two dimensional component Σ of the fixed point set there is a **fat** vertex $\langle \Sigma \rangle$ labeled by two real numbers and one integer: the **momentum map label** $\Phi(\Sigma)$, the **area label** $\frac{1}{2\pi} \int_{\Sigma} \omega$, and the **genus** g of the surface Σ . A **\mathbf{Z}_ℓ -sphere** is a 2-sphere in M on which the circle acts by rotations of speed ℓ . For each \mathbf{Z}_ℓ -sphere containing two fixed points p and q , the graph has an edge connecting the vertices $\langle p \rangle$ and $\langle q \rangle$ labeled by the integer ℓ ; the difference $|\Phi(p) - \Phi(q)|$ is equal to the symplectic area of the sphere times $\frac{\ell}{2\pi}$. Note that there is an isotropy weight ℓ at the south pole of the \mathbf{Z}_ℓ sphere, and weight $-\ell$ at the north pole. The decorated graphs for two different circle actions on $\mathbb{C}P^2$ are shown in Figure 2.1.

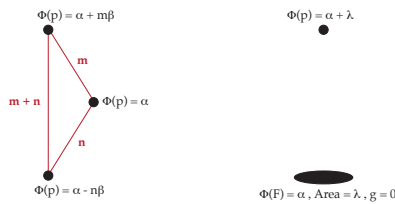


FIGURE 2.1. The decorated graphs for $S^1 \curvearrowright \mathbb{C}P^2$: with isolated fixed points on the left; and with a fixed surface on the right.

When the fixed points of the action $S^1 \curvearrowright M^4$ are isolated, the S^1 -action extends to a toric action $T^2 \curvearrowright M^4$. If there is a single critical surface Σ , then we may deduce that Σ has genus 0, and $S^1 \curvearrowright M^4$ is either a k -fold S^1 -equivariant symplectic blowup of the projective plane, or one of the two S^2 bundles over S^2 : the trivial $\mathbb{C}P^1 \times \mathbb{C}P^1$ or the non-trivial $M_{\mathbb{C}P^1}$, with some Hamiltonian circle action. Finally, when there are two fixed surfaces Σ_{min} and Σ_{max} , they must have the same genus, so are each homeomorphic to a fixed surface Σ . Moreover we can construct $S^1 \curvearrowright M$ as a k -fold S^1 -equivariant symplectic blowup of $\Sigma \times S^2$ or of M_Σ , the non-trivial S^2 -bundle over Σ , with some Hamiltonian S^1 -action. We call the case when there are two fixed surfaces of genus $g > 0$ the **positive genus** case.

In what follows, we will use the following notation for a Hamiltonian $S^1 \curvearrowright M^4$. Let $\Phi : M \rightarrow \mathbb{R}$ be the momentum map. Let B_{min} and B_{max} be the extremal critical sets of Φ . For $* = min, max$, we define

$$s_* = \int_{B_*} \omega ; y_* = \Phi(B_*) ; \text{ and } e_* = \begin{cases} B_* \cdot B_* & \text{when } \dim(B_*) = 2 \\ \frac{1}{mn} & \text{when } \dim(B_*) = 0 \end{cases} ,$$

where m and n are the isotropy weights at B_* when it is an isolated fixed point. For an interior isolated fixed point p , we define $y_p = \Phi(p)$, and let m_p and n_p be the absolute values of the isotropy weights at p . We let $e_p = \frac{1}{m_p n_p}$.

These parameters are related by the following formulæ. The proof of the main theorem does not depend on them, but to do any computation, they are essential.

$$(2.2) \quad e_{min} = \frac{\left(\sum_p y_p e_p\right) + s_{min} - \left(\sum_p e_p\right) \cdot y_{max} - s_{max}}{y_{max} - y_{min}}$$

and

$$(2.3) \quad e_{max} = \frac{\left(\sum_p e_p\right) \cdot y_{min} + s_{max} - \left(\sum_p y_p e_p\right) - s_{min}}{y_{max} - y_{min}} ,$$

where p runs over the interior fixed points. Formulæ (2.2) and (2.3) can be deduced from [8, Proof of Lemma 2.18], which has a missing term that we have restored (the missing term is the s_{max} ; its absence does not affect the validity of Karshon's proof).

3. EQUIVARIANT COHOMOLOGY

In this paper, we are interested in computing the (Borel) equivariant cohomology of a Hamiltonian S^1 -space. Equivariant cohomology is a generalized cohomology theory in the equivariant category. That is, it satisfies the usual axioms we expect of cohomology for spaces equipped with group actions, together with equivariant maps between such spaces. That the cohomology theory is “generalized” means that the equivariant cohomology of a point is richer than just a copy of the coefficient ring.

In the case of circle actions, we define the classifying bundle $ES^1 := S^\infty$ to be the unit sphere in an infinite dimensional complex Hilbert space \mathbb{C}^∞ . This space is contractible and equipped with a free S^1 -action by coordinate multiplication. We then define

$$H_{S^1}^*(M; R) = H^*((M \times ES^1)/S^1; R),$$

where $S^1 \curvearrowright (M \times ES^1)$ diagonally and R is the coefficient ring. The classifying space is $BS^1 = ES^1/S^1 = \mathbb{C}P^\infty$. The equivariant cohomology of a point, then, is

$$(3.1) \quad H_{S^1}^*(pt; R) = H^*(BS^1; R) = H^*(\mathbb{C}P^\infty; R) = R[u],$$

where $\deg(u) = 2$. More generally for a torus $T = (S^1)^k$, we have

$$H_{T^k}^*(M; R) = H^*((M \times (ES^1)^k)/T^k; R),$$

and in particular,

$$H_T^*(pt; R) = H^*((\mathbb{C}P^\infty)^k; R) = R[u_1, \dots, u_k],$$

where $\deg(u_i) = 2$.

If we endow a point pt with the trivial T -action, then the constant map

$$\pi : M \rightarrow pt$$

is equivariant. This induces a map in equivariant cohomology $\pi^* : H_T^*(pt; R) \rightarrow H_T^*(M; R)$ which endows $H_T^*(M; R)$ with a $H_T^*(pt; R)$ -module structure. In the context of Hamiltonian torus actions, Ginzburg studied the $H_T^*(pt; R)$ -module structure for coefficient rings which are fields of characteristic 0. He proved the following, adapted to our context.

3.2. Theorem (Ginzburg [4, Cor. 3.4]). *Let $T \curvearrowright M$ be a compact Hamiltonian T -space. If \mathbb{F} is a field of characteristic zero, then $H_T^*(M; \mathbb{F})$ is a free $H_T^*(pt; \mathbb{F})$ -module isomorphic to $H^*(M; \mathbb{F}) \otimes H_T^*(pt; \mathbb{F})$.*

We also have the inclusion of the fixed point set, $i : M^T \hookrightarrow M$. This is an equivariant map, and Borel studied the induced map in equivariant cohomology.

3.3. Theorem (Borel [3]). *Let a torus T act on a compact manifold M . In equivariant cohomology, the kernel and cokernel of the map induced by inclusion,*

$$i^* : H_T^*(M; R) \rightarrow H_T^*(M^T; R)$$

are torsion submodules. In particular, if $H_T^(M^T; R)$ is torsion free, then i^* is injective.*

There are a number of tools to describe the image of i^* . The **one-skeleton** of a torus $T = (S^1)^k$ action on M is the set $M_{(1)} = \{x \in M \mid \dim(T \cdot x) \leq 1\}$. Tolman and Weitsman considered the equivariant map, the inclusion of the fixed points $j : M^T \rightarrow M_{(1)}$, and they proved the following theorem.

3.4. Theorem (Tolman-Weitsman [12]). *Let M be a compact Hamiltonian T -space. The induced maps in equivariant cohomology with rational coefficients,*

$$\begin{array}{ccc} H_T^*(M; \mathbb{Q}) & & H_T^*(M_{(1)}; \mathbb{Q}) \\ & \searrow i^* & \swarrow j^* \\ & H_T^*(M^T; \mathbb{Q}) & \end{array}$$

have the same image in $H_T^(M^T; \mathbb{Q})$.*

The Atiyah-Bott/ Berline-Vergne (ABBV) localization formula [1, 2] expresses the integral over M of an equivariant cohomology class as a sum of integrals over the connected components F of the fixed point set M^T as follows.

3.5. Theorem (Atiyah-Bott [1] / Berline-Vergne [2]). *Suppose a compact torus T acts on a compact manifold M . Then for any class $\alpha \in H_T^*(M; \mathbb{Q})$,*

$$(3.6) \quad \pi_*(\alpha) = \sum_{F \subseteq M^T} \pi_*^F \left(\frac{\alpha|_F}{e_T(\nu(F \subseteq M))} \right).$$

where the sum on the right-hand side is taken over the connected components F of the fixed point set M^T , $\alpha|_F$ is the restriction of α to F , and $e_T(\nu(F \subseteq M))$ is the equivariant Euler class of the normal bundle of F ; the map π is the equivariant pushforward $M \rightarrow \text{pt}$, and $\pi^F : F \rightarrow \text{pt}$ is the pushforward of F to a point.

4. THE CIRCLE EQUIVARIANT COHOMOLOGY OF A HAMILTONIAN 4-MANIFOLD

Proof of Part (A) of Theorem 1.1. We consider a Hamiltonian S^1 -action on a four-manifold M . The fixed point set M^{S^1} consists of isolated points and up to two surfaces. The surfaces are symplectic submanifolds and are hence orientable. Thus, $H^*(M^{S^1}; \mathbb{Z})$ is torsion free. It now follows from Theorem 3.3 that $i^* : H^*(M; \mathbb{Z}) \rightarrow H^*(M^{S^1}; \mathbb{Z})$ is injective.

For the remainder of the section, we work with the coefficient ring \mathbb{Q} . The results hold over any field of characteristic zero.

Deducing the ranks of $i^*(H_{S^1}^*(M; \mathbb{Q}))$ from equivariant Poincaré polynomials.

Let M be a compact symplectic manifold with a Hamiltonian circle action. Theorem 3.2 guarantees that the S^1 -equivariant cohomology of M splits

$$H_{S^1}^*(M; \mathbb{Q}) \cong H^*(M; \mathbb{Q}) \otimes H_{S^1}^*(pt; \mathbb{Q}),$$

as $H_{S^1}^*(pt; \mathbb{Q})$ -modules. Hence the equivariant Poincaré polynomial splits:

$$(4.1) \quad P_{S^1}^M(t) = P^M(t) \cdot P_{S^1}^{pt}(t).$$

By (3.1),

$$(4.2) \quad P_{S^1}^{pt}(t) = (1 + t^2 + t^4 + \dots) = \frac{1}{1 - t^2}.$$

We use Morse theory to find the Poincaré polynomial $P^M(t)$. The momentum map of the Hamiltonian circle action is a perfect Morse-Bott function whose critical points are the fixed points for the circle action [7, §32]. Therefore

$$(4.3) \quad \dim(H^j(M; \mathbb{Q})) = \sum \dim H^{j-\lambda_F}(F; \mathbb{Q}),$$

where we sum over the connected components F of the fixed point set, and where λ_F is the index of the component F .

In the special case of $S^1 \curvearrowright M^4$, the fixed point components are finitely many isolated points and up to two surfaces, of the same genus g . By [9], the contribution of each fixed point component is listed in the table below.

F	$\Phi(F)$	λ_F	Contribution to				
			$H^0(M)$	$H^1(M)$	$H^2(M)$	$H^3(M)$	$H^4(M)$
fixed surface	minimal	0	1	$2g$	1		
	maximal	2			1	$2g$	1
isolated fixed point	minimal	0	1				
	interior	2			1		
	maximal	4					1

TABLE 4.4. Table of contributions to $\dim H^j$, from [9]

Hence

$$(4.5) \quad P^M(t) = 1 + \delta_{min}2gt + (\ell - 2 + 2\delta_{min} + 2\delta_{max})t^2 + \delta_{max}2gt^3 + t^4,$$

and

$$(4.6) \quad \begin{aligned} P^{M^{S^1}}(t) &= |\text{isolated points in } M^{S^1}| + |\text{surfaces in } M^{S^1}|(1 + 2gt + t^2) \\ &= (\ell + \delta_{min} + \delta_{max}) + (\delta_{min} + \delta_{max})2gt + (\delta_{min} + \delta_{max})t^2, \end{aligned}$$

where

$$\ell = \# \text{ isolated fixed points,}$$

$$\delta_{min} = \# \text{ minimal fixed surfaces (zero or one), and}$$

$$\delta_{max} = \# \text{ maximal fixed surfaces (zero or one).}$$

Therefore

$$(4.7) \quad \begin{aligned} P_{S^1}^M(t) &= P^M(t) \cdot \frac{1}{1-t^2} \\ &= 1 + (\ell - 1 + 2\delta_{min} + 2\delta_{max})t^2 + (\ell + 2\delta_{min} + 2\delta_{max})t^4 \left(\frac{1}{1-t^2}\right) \\ &\quad + \delta_{min}2gt + (\delta_{min} + \delta_{max})2gt^3 \left(\frac{1}{1-t^2}\right), \end{aligned}$$

and

$$\begin{aligned}
 P_{S^1}^{M^{S^1}}(t) &= P^{M^{S^1}}(t) \cdot \frac{1}{1-t^2} \\
 (4.8) \quad &= (\ell + \delta_{\min} + \delta_{\max}) + (\ell + 2\delta_{\min} + 2\delta_{\max})t^2 \\
 &\quad + (\ell + 2\delta_{\min} + 2\delta_{\max})t^4 \left(\frac{1}{1-t^2}\right) \\
 &\quad + (\delta_{\min} + \delta_{\max})2gt + (\delta_{\min} + \delta_{\max})2gt^3 \left(\frac{1}{1-t^2}\right).
 \end{aligned}$$

The differences between the corresponding coefficients in $P_{S^1}^{M^{S^1}}(t)$ and $P_{S^1}^M(t)$ tell us how many constraints cut out $i^*(H_{S^1}^*(M; \mathbb{Q}))$ in $H_{S^1}^*(M^{S^1}; \mathbb{Q})$. The constraints are linear relations among the equivariant cohomology classes on M^{S^1} , and we will refer colloquially to the **relations** the classes must satisfy. Equations (4.7) and (4.8) combine to give the following lemma.

4.9. Lemma. *Let $S^1 \curvearrowright M^4$ be a compact Hamiltonian space.*

$$\begin{aligned}
 P_{S^1}^{M^{S^1}}(t) - P_{S^1}^M(t) &= \left[(\ell + \delta_{\max} + \delta_{\min} - 1) + \delta_{\max}2gt + (2 - \ell - \delta_{\max} - \delta_{\min})t^2 \right. \\
 &\quad \left. - \delta_{\max}2gt^3 - t^4 \right] \cdot (1 + t^2 + t^4 + \dots) \\
 &= (\# \text{ components of } M^{S^1} - 1) + 2gt + t^2.
 \end{aligned}$$

The coefficient $2g$ of t in the last equality follows because if $g > 0$, we must have $\delta_{\max} = 1$. Hence we must find $(\# \text{ components of } M^{S^1} - 1)$ relations in degree 0; $2g$ relations in degree 1; and one relation in degree 2 to determine the image $i^*(H_{S^1}^*(M; \mathbb{Q})) \subset H_{S^1}^*(M^{S^1}; \mathbb{Q})$.

Calculations of equivariant Euler classes and their inverses. To interpret the ABBV relation we calculate explicitly equivariant Euler classes and their inverses. For the Euler classes, we work with integer coefficients. For their inverses, we must revert to \mathbb{Q} . In the case of an equivariant bundle over a point, applying the splitting principle in equivariant cohomology [6, Theorem C.31], the formula [6, (C.13)] simplifies to the following single term.

4.10. Lemma. *Consider a linear circle action $S^1 \curvearrowright \mathbb{C}^n$ with weights $b_1, \dots, b_n \in \mathbb{Z}$. Thought of as an equivariant bundle over a point $\mathbb{C}^n = \nu(\{\vec{0}\} \subset \mathbb{C}^n) \rightarrow \vec{0}$, this has equivariant Euler class*

$$e_{S^1}(\mathbb{C}^n) = (-1)^n b_1 \cdots b_n u^n \in H_{S^1}^*(pt; \mathbb{Z}) = \mathbb{Z}[u],$$

with (formal) inverse

$$(e_{S^1}(\mathbb{C}^n))^{-1} = \frac{(-1)^n}{b_1 \cdots b_n u^n} \in \mathbb{Q}[u, u^{-1}].$$

In the case of an equivariant complex line bundle over a positive-dimensional manifold, where the action fixes the zero-section, we may also identify the equivariant Euler class explicitly. Moreover, in this case, the equivariant Euler class is invertible (in the appropriate ring), and we have an explicit formula for its inverse.

4.11. **Lemma.** *Let*

$$S^1 \curvearrowright \left(\begin{array}{c} \mathcal{L} \\ \downarrow \\ \Sigma \end{array} \right)$$

be an equivariant complex line bundle with fixed set precisely the zero section. At any point $p \in \Sigma$, let $b \in \mathbb{Z}$ denote the weight of the circle action on the fibre over p . Then the equivariant Euler class of \mathcal{L} has the form

$$(4.12) \quad e_{S^1}(\mathcal{L}) = -1 \otimes b \cdot u + e(\mathcal{L}) \otimes 1 \in H_{S^1}^2(\Sigma; \mathbb{Z}),$$

where $e(\mathcal{L}) \in H^2(\Sigma; \mathbb{Z})$ denotes the ordinary Euler class of \mathcal{L} . Its inverse (in the ring of rational functions with coefficients in $H^(\Sigma; \mathbb{Q})$, namely $H^*(\Sigma; \mathbb{Q})[u, u^{-1}]$) is*

$$(4.13) \quad e_{S^1}(\mathcal{L})^{-1} = - \sum_{i=0}^N e(\mathcal{L})^i \otimes \left(\frac{1}{bu} \right)^{i+1},$$

where $N = \left\lfloor \frac{\dim(\Sigma)}{2} \right\rfloor$.

Proof. We first note that because the S^1 action fixes Σ , we have the splitting

$$H_{S^1}^*(\Sigma; \mathbb{Z}) = H^*(\Sigma; \mathbb{Z}) \otimes H_{S^1}^*(pt; \mathbb{Z}).$$

Moreover,

$$e_{S^1}(\mathcal{L}) \in H_{S^1}^2(\Sigma; \mathbb{Z})$$

and by the splitting

$$H_{S^1}^2(\Sigma; \mathbb{Z}) \cong \left(H^0(\Sigma; \mathbb{Z}) \otimes H_{S^1}^2(pt; \mathbb{Z}) \right) \oplus \left(H^2(\Sigma; \mathbb{Z}) \otimes H_{S^1}^0(pt; \mathbb{Z}) \right).$$

The leading term in (4.12) is guaranteed by [6, (C.13)]. Furthermore, the equivariant Euler class is defined to be the Euler class of the pull-back bundle

$$\begin{array}{ccc} p^* \mathcal{L} & \xrightarrow{\quad \quad} & \mathcal{L} \\ \downarrow & & \downarrow \\ \Sigma \times_{S^1} ES^1 & \xrightarrow[p]{} & \Sigma. \end{array}$$

By naturality of characteristic classes, we must have that the restriction

$$p^*(e_{S^1}(\mathcal{L})) = e(\mathcal{L}),$$

and so the second term in (4.12) must be $e(\mathcal{L}) \otimes 1$.

To check our formula for $e_{S^1}(\mathcal{L})^{-1}$, we take the product

$$\begin{aligned} (-1 \otimes b \cdot u + e(\mathcal{L}) \otimes 1) &\cdot \left(- \sum_{i=0}^N e(\mathcal{L})^i \otimes \left(\frac{1}{bu} \right)^{i+1} \right) \\ &= \sum_{i=0}^N e(\mathcal{L})^i \otimes \left(\frac{1}{bu} \right)^i - \sum_{i=0}^N e(\mathcal{L})^{i+1} \otimes \left(\frac{1}{bu} \right)^{i+1} \\ &= 1 \otimes 1 - e(\mathcal{L})^{N+1} \otimes \left(\frac{1}{bu} \right)^{N+1}. \end{aligned}$$

But $e(\mathcal{L})^{N+1} = 0$ for dimension reasons, so we see that the product equals $1 \otimes 1$, as desired. \square

Now let $S^1 \curvearrowright M^4$ be a Hamiltonian action. For an isolated fixed point $p \in M^{S^1}$, the equivariant Euler class is an element of $H_{S^1}^4(p; \mathbb{Z})$. By Lemma 4.10,

$$e_{S^1}(\nu(\{p\} \subset M)) = -m_p n_p u^2$$

with inverse

$$(4.14) \quad (e_{S^1}(\nu(\{p\} \subset M)))^{-1} = -e_p \frac{1}{u^2}.$$

For an S^1 -fixed surface Σ (which must be a minimum or maximum critical set), the equivariant Euler class is an element of $H_{S^1}^2(\Sigma; \mathbb{Z})$. For any point $p \in \Sigma$, the S^1 -weight in the normal direction to Σ is ± 1 . It must be so because the action is effective and if it were $\pm b$, there would be a global \mathbf{Z}_b stabilizer. Moreover, it is positive when Σ is a minimum and negative when Σ is a maximum. From Lemma 4.11, then, the equivariant Euler class is

$$e_{S^1}(\nu(\Sigma \subset M)) = \pm 1 \otimes u + e_\Sigma[\Sigma] \otimes 1,$$

where the first sign is determined by whether Σ is a minimum ($-$) or maximum ($+$), and e is the self-intersection $\Sigma \cdot \Sigma$. Under the identification $H^2(\Sigma; \mathbb{Z}) \cong \mathbb{Z}$, the self intersection is the ordinary Euler class of the normal bundle $\nu(\Sigma \subset M)$. It satisfies the formulæ (2.2) and (2.3). By Lemma 4.11, the inverse is

$$(4.15) \quad e_{S^1}(\nu(\Sigma \subset M))^{-1} = \pm 1 \otimes \frac{1}{u} - e_\Sigma[\Sigma] \otimes \frac{1}{u^2}.$$

Consequence of the ABBV relation (1.2). The ABBV relation will impose one constraint in homogeneous degree 2 on tuples

$$\alpha = (\alpha|_F) \in H_{S^1}^*(M^{S^1}; \mathbb{Q}) = \bigoplus_{F \subset M^{S^1}} H_{S^1}^*(F; \mathbb{Q}).$$

Suppose we have such a tuple $\alpha = (\alpha|_F)$. At each isolated fixed point p , we may identify $H_{S^1}^2(p; \mathbb{Q}) = H^0(p; \mathbb{Q}) \otimes H_{S^1}^2(pt; \mathbb{Q})$. Thus, $\alpha|_p = 1 \otimes c_p u$ for some $c_p \in \mathbb{Q}$. In the ABBV

relation (1.2), this will contribute

$$\pi_*^p \left[\alpha|_p \cdot (e_{S^1}(\nu(p \subset M)))^{-1} \right] = \pi_*^p \left[(1 \otimes c_p u) \cdot \left(-e_p \otimes \frac{1}{u^2} \right) \right] = -\frac{c_p e_p}{u},$$

where the first equality is by (4.14). Next, for a fixed surface Σ ,

$$H_{S^1}^2(\Sigma; \mathbb{Q}) = (H^2(\Sigma; \mathbb{Q}) \otimes H_{S^1}^0(pt; \mathbb{Q})) \oplus (H^0(\Sigma; \mathbb{Q}) \otimes H_{S^1}^2(pt; \mathbb{Q})).$$

Thus $\alpha|_\Sigma = [\Sigma] \otimes a_\Sigma + b_\Sigma \otimes u$, where $a_\Sigma, b_\Sigma \in \mathbb{Q}$. In the term in (1.2), this will contribute

$$\begin{aligned} \pi_*^\Sigma \left[\alpha|_\Sigma \cdot (e_{S^1}(\nu(\Sigma \subset M)))^{-1} \right] &= \pi_*^\Sigma \left[([\Sigma] \otimes a_\Sigma + b_\Sigma \otimes u) \cdot (e_{S^1}(\nu(\Sigma \subset M)))^{-1} \right] \\ &= \pi_*^\Sigma \left[([\Sigma] \otimes a_\Sigma + b_\Sigma \otimes u) \cdot \left(\pm 1 \otimes \frac{1}{u} - e_\Sigma[\Sigma] \otimes \frac{1}{u^2} \right) \right] \\ &= \pm \frac{a_\Sigma}{u} - \frac{b_\Sigma e_\Sigma}{u}, \end{aligned}$$

where the second equality is by (4.15). Combining these, we get a term of the form

$$\sum_p -\frac{c_p e_p}{u} + \delta_{max} \left(\frac{a_{max}}{u} - \frac{b_{max} e_{max}}{u} \right) - \delta_{min} \left(\frac{a_{min}}{u} + \frac{b_{min} e_{min}}{u} \right),$$

where p runs over all isolated fixed points. This is in $\mathbb{Q}[u]$ if and only if

$$(4.16) \quad \left(\sum_p -c_p e_p \right) + \delta_{max} (a_{max} - b_{max} e_{max}) - \delta_{min} (a_{min} + b_{min} e_{min}) = 0.$$

This precisely gives us one linear relation among the rational numbers $c_p, a_{max}, b_{max}, a_{min}$ and b_{min} .

Proof of Part (B) of Theorem 1.1. We want to determine which classes in $H_{S^1}^*(M^{S^1}; \mathbb{Q})$ are the images of global equivariant classes. Let S denote the submodule of classes in $H_{S^1}^*(M^{S^1}; \mathbb{Q})$ which satisfy conditions (0), (1) and (2) of Theorem 1.1. As a submodule of a free module over the PID $\mathbb{Q}[u] = H_{S^1}^*(pt; \mathbb{Q})$, the submodule S is itself a free $H_{S^1}^*(pt; \mathbb{Q})$ -module.

By Part (A) of Theorem 1.1 and Theorem 3.2, we also know that $i^*(H_{S^1}^*(M; \mathbb{Q}))$ is a free submodule of $H_{S^1}^*(M^{S^1}; \mathbb{Q})$. We aim to show that $i^*(H_{S^1}^*(M; \mathbb{Q})) \subset S$ and that these have equal rank in homogeneous degree k for each k . This will prove that $i^*(H_{S^1}^*(M; \mathbb{Q})) = S$.

We first consider equivariant cohomology classes of homogeneous degree zero. By Theorem 3.2, we have $H_{S^1}^0(M; \mathbb{Q}) = H^0(M) \otimes H_{S^1}^0(pt; \mathbb{Q}) = \mathbb{Q} \otimes H_{S^1}^0(pt; \mathbb{Q})$, and so

$$\dim(H_{S^1}^0(M; \mathbb{Q})) = 1$$

over $H_{S^1}^0(pt; \mathbb{Q})$. Constant functions on M are equivariant, so they represent classes in $H_{S^1}^0(M; \mathbb{Q})$. They must represent all of $H_{S^1}^0(M; \mathbb{Q})$ since it is one dimensional. Thus, for $\alpha \in i^*(H_{S^1}^0(M; \mathbb{Q}))$, its restriction to any fixed component is its constant value. This means that for a class in $H_{S^1}^0(M^{S^1}; \mathbb{Q})$ to be in the image of i^* , it must be a constant

tuple, which is equivalent to satisfying $(\# \text{ components of } M^{S^1} - 1)$ relations which force the tuple to be constant. These are the $(\# \text{ components of } M^{S^1} - 1)$ relations sought in Lemma 4.9.

Next, we note that because the action $S^1 \curvearrowright M^{S^1}$ is trivial,

$$H_{S^1}^1(M^{S^1}; \mathbb{Q}) = \left(H^1(M^{S^1}; \mathbb{Q}) \otimes H_{S^1}^0(pt; \mathbb{Q}) \right) \oplus \left(H^0(M^{S^1}; \mathbb{Q}) \otimes H_{S^1}^1(pt; \mathbb{Q}) \right),$$

and the second term on the right-hand side is zero since BS^1 is simply connected. Moreover, $H^1(M^{S^1})$ is non-zero only if there are fixed surfaces of positive genus. Thus, $H_{S^1}^1(M^{S^1})$ is non-zero only in the positive genus case. In that case, we have two fixed surfaces of the same genus. A homogeneous equivariant class of degree 1 will be zero on each interior fixed point. A globally constant class of homogeneous degree one is, as ever, S^1 -equivariant. Such a class will restrict to the same class on Σ_{max} and Σ_{min} . That is, we will have a pair $(\alpha|_{\Sigma_{min}}, \alpha|_{\Sigma_{max}})$ for which, when we identify $H^1(\Sigma_{min}; \mathbb{Q}) \cong H^1(\Sigma_{max}; \mathbb{Q})$, we have $\alpha|_{\Sigma_{min}} = \alpha|_{\Sigma_{max}}$. The possible classes of this form make up a $\dim(H^1(\Sigma; \mathbb{Q})) = 2g$ dimensional subspace of $H_{S^1}^1(M^{S^1})$. We know from (4.7) that $i^*(H_{S^1}^1(M; \mathbb{Q}))$ is $2g$ dimensional, so as in the degree zero case, these must be everything in the image of i^* . In terms of relations, we will have exactly the $4g - 2g = 2g$ relations that force $\alpha|_{\Sigma_{min}} = \alpha|_{\Sigma_{max}}$, namely the $2g$ relations sought in Lemma 4.9.

We conclude that $i^*(H_{S^1}^*(M; \mathbb{Q}))$ is a subset of the submodule of classes in $H_{S^1}^*(M^{S^1}; \mathbb{Q})$ which satisfy conditions (0) and (1). By the ABBV localization formula 3.5, every class in $i^*(H_{S^1}^*(M; \mathbb{Q}))$ is in the submodule of classes that satisfy the ABBV relation (1.2). Therefore, $i^*(H_{S^1}^*(M; \mathbb{Q}))$ is a subset of the intersection submodule S . The ABBV relation imposes weaker constraints than being globally constant in homogenous degree zero, and it imposes no constraints in homogenous degree one. In homogeneous degree two, it imposes exactly one constraint (4.16). This is precisely the one degree two relation sought in Lemma 4.9.

Thus, we have verified that $i^*(H_{S^1}^*(M; \mathbb{Q})) \subset S$ and has the same ranks, so the two (free) submodules must be equal. This completes the proof of Theorem 1.1. \square

To assemble the equivariant cohomology of a complexity one space, we will need a slightly more general form of Theorem 1.1. We consider a Hamiltonian T -action on a symplectic four-manifold M which is the extension of a Hamiltonian $S^1 \curvearrowright M$ by a trivial action of a subtorus K of codimension one. This forces the fixed point set M^T to consist of isolated points and up to two surfaces. We still have the parameters associated to the decorated graph described in Section 2 for the Hamiltonian T/K -action.

4.17. Proposition. *Let M be a compact connected symplectic four-manifold. Let a torus T act non-trivially in a Hamiltonian fashion on M , and suppose that a codimension one subtorus $K \subset T$ acts trivially. Let $\pi_K : H_T^*(-; \mathbb{Q}) \rightarrow H_K^*(-; \mathbb{Q})$ be the restriction map in equivariant cohomology.*

(A) The inclusion $i : M^T \hookrightarrow M$ induces an injection in integral equivariant cohomology

$$i^* : H_T^*(M; \mathbb{Z}) \hookrightarrow H_T^*(M^T; \mathbb{Z}).$$

(B) In equivariant cohomology with rational coefficients, the image of i^* is characterized as those classes $\alpha \in H_T^*(M^T; \mathbb{Q})$ which satisfy:

- (1) $\pi_K(\alpha|_{F_i}) = \pi_K(\alpha|_{F_j})$ for all components F_i, F_j of M^T ; and
- (2) the ABBV relation

$$(4.18) \quad \sum_{F \subset M^T} \pi_*^F \left(\frac{\alpha|_F}{e_{T/K}(\nu(F \subseteq M))} \right) \in H_T^*(pt; \mathbb{Q}),$$

where the sum is taken over connected components F of the fixed point set M^T , $\alpha|_F$ is the restriction of α to the component F , π_*^F is the equivariant pushforward map, and the equivariant Euler classes are taken with respect to the effective T/K -action.

The proof is identical to the proof of Theorem 1.1, but the conditions (0) and (1) in that theorem have become the single more compact form (1) in this generalization. Writing the conditions in this way is equivalent to saying that $\alpha|_{F_i} - \alpha|_{F_j}$ is in $\ker(\pi_K)$. This boils down to requiring that $\alpha|_{F_i} - \alpha|_{F_j}$ is a multiple of $1 \otimes \tau$, where τ is a generator for the dual of the Lie algebra of T/K . Since the generator τ is an element of $H_{T/K}^2(pt; \mathbb{Q})$, this imposes the requirement in degrees zero and one that the classes $\alpha|_{F_i}$ and $\alpha|_{F_j}$ agree.

4.19. Remark. When the fixed point set consists of isolated points, Theorem 1.1 coincides with [5, Proposition 3.1], and Proposition 4.17 coincides with [5, Proposition 3.2]. In the case when there are fixed surfaces with genus 0, our results are already providing new calculations.

5. APPLICATIONS TO COMPLEXITY ONE SPACES

Recall that a complexity one space is a symplectic manifold equipped with the action of a torus which is one dimension less than half the dimension of the manifold. That is, the torus is one dimension too small for the action to be toric. These manifolds have been classified in terms of combinatorial and homotopic data, together called a **painting**, by Karshon and Tolman [10].

Suppose that $T^{n-1} \curvearrowright M^{2n}$ is a complexity one space. Then for any subtorus $T^k \subset T^{n-1}$, the set of fixed points M^{T^k} is a symplectic submanifold with components of dimension at most $2n - 2k$. Moreover, $(T^{n-1}/T^k) \curvearrowright M^{T^k}$ is an effective action, so the components of M^{T^k} have dimension at least $2n - 2k - 2$. Thus, M^{T^k} consists of a collection of toric pieces and complexity one pieces. In particular, the components of the one-skeleton are two-dimensional or four-dimensional. Tolman and Weitsman's Theorem 3.4 says that for a tuple of equivariant classes on the fixed point components to be in the image of i^* , we only need to ensure that the tuple is a global class on each component of the

one-skeleton. Thus combining Theorem 3.4 with our result in Proposition 4.17 gives the following combinatorial description of the equivariant cohomology of a complexity one space.

5.1. Corollary. *Let $T^{n-1} \curvearrowright M^{2n}$ be a compact, connected complexity one space.*

(A) *The inclusion $i : M^T \hookrightarrow M$ induces an injection in integral equivariant cohomology*

$$i^* : H_T^*(M; \mathbb{Z}) \hookrightarrow H_T^*(M^T; \mathbb{Z}).$$

(B) *In equivariant cohomology with rational coefficients, the image of i^* is characterized as those classes $\alpha \in H_T^*(M^T; \mathbb{Q})$ which satisfy, for every codimension one subtorus $K \subset T$ and every connected component X of M^K ,*

- (1) $\pi_K(\alpha|_{F_i}) = \pi_K(\alpha|_{F_j})$ for all components F_i, F_j of X^T ; and
- (2) when $\dim(X) = 4$, the ABBV relation,

$$(5.2) \quad \sum_{F \subset X^T} \pi_*^F \left(\frac{\alpha|_F}{e_{T/K}(\nu(F \subseteq X))} \right) \in H_T^*(pt; \mathbb{Q}),$$

where the sum is taken over connected components F of the fixed point set X^T , $\alpha|_F$ is the restriction of α to the component F , π_*^F is the equivariant pushforward map, and the equivariant Euler classes are taken with respect to the effective T/K -action.

We conclude with an example of a complexity one space and indicate some of the computations our work allows.

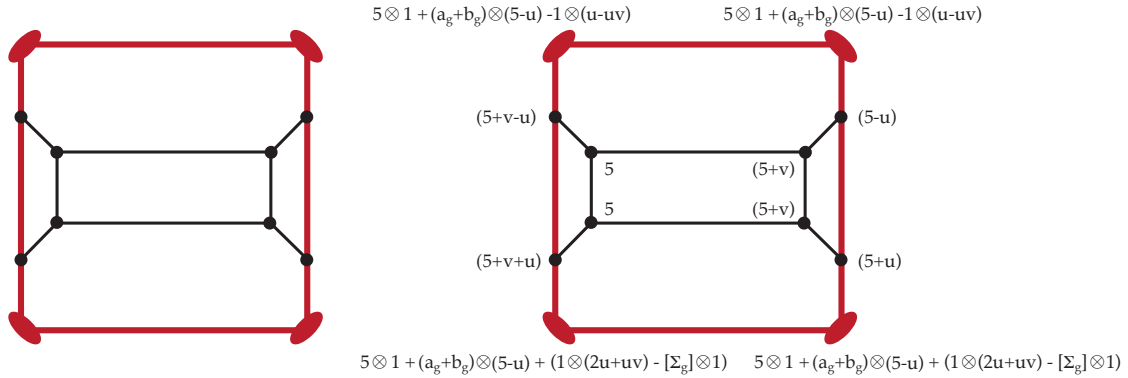


FIGURE 5.3. On the left, the x-ray for a complexity one space for a T^2 acting on M^6 . The red fat vertices correspond to genus g surfaces fixed by T . The black vertices correspond to isolated fixed points. The red edges correspond to four-manifolds fixed by a circle, and the black edges correspond to 2-spheres fixed by a circle. This manifold has Betti numbers $\beta_0 = \beta_6 = 1$, $\beta_1 = \beta_5 = g$, $\beta_2 = \beta_4 = 7$ and $\beta_3 = 2g$. On the right, a collection of classes in $H_T^*(F) = H^*(F)[u, v]$ for each fixed component F . These classes satisfy the requirements in Corollary 5.1, so they are the restrictions to the fixed sets of a global class in $H_T^*(M; \mathbb{Q})$.

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